ON THE MINIMAL SPACE PROBLEM AND A NEW RESULT ON EXISTENCE OF BASIC SEQUENCES IN QUASI-BANACH SPACES

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Dedicated to Nigel J. Kalton

Abstract. We prove that if X is a quasi-normed space which possesses an infinite countable dimensional subspace with a separating dual, then it admits a strictly weaker Hausdorff vector topology. Such a topology is constructed explicitly. As an immediate consequence, we obtain an improvement of a well-known result of Kalton-Shapiro [13] and Drewnowski [6, 7] by showing that a quasi-Banach space contains a basic sequence if and only if it contains an infinite countable dimensional subspace whose dual is separating. We also use this result to highlight a new feature of the minimal quasi-Banach space constructed by Kalton [15]. Namely, which all of its $\aleph_0$-dimensional subspaces fail to have a separating family of continuous linear functionals.

Keywords: Quasi-normed spaces, minimal spaces, weaker Hausdorff vector topologies, basic sequences.

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1. Introduction

The notion of a minimal topological vector space was introduced by Nigel Kalton [12]. The minimal space problem asks whether a given F-space (i.e., complete linear metric space) is or not minimal. Recall a topological vector space (X, τ) is called minimal if for every Hausdorff vector topology $\rho \leq \tau$ we have $\rho = \tau$. It is proved in [12] that the space ω of all sequences of real numbers is minimal. Key examples of non-minimal spaces are the Banach spaces and the non-locally convex space $M[0,1]$ of all measurable functions on $[0,1]$ with its standard F-norm (cf. [21, Theorem 1.1]). We would like to refer the reader the works of Drewnowski [6, 7] for more information on minimal spaces and their correlates. Besides their own importance, minimal spaces have high potential of applications in many situations (see, e.g., [3, 5, 8, 11, 12, 14, 16, 17, 21] and the references quoted there). Lately, they have also been especially important for complex interpolation (cf. [10, pp. 121–177]). The purpose of this paper is to provide a new criterion for a quasi-normed space to be non-minimal. Precisely, the main result of this paper is as follows.

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Main Theorem. *Let X be a quasi-normed space. Assume that X contains an infinite countable dimensional subspace whose dual is separating. Then X is non-minimal.*

The motivation for this result comes mainly from the well-established link between minimal spaces and the famous basic sequence problem. Here, we only highlight a few aspects of the link. In their seminal works, Kalton [12] and Kalton-Shapiro [13] provided fundamental techniques for building basic sequences in non-minimal F-spaces. Among other, they proved that an F-space X contains a basic sequence iff it admits a strictly weaker Hausdorff vector topology. What is more, the containement of basic sequences is equivalent to the existence of closed infinite-dimensional subspaces with separating dual (see [12, theorems 4.2 and 4.4] and [13, theorems 2.1, 3.2, and Corollary 3.3]). Just for the especial case of the quasi-Banach space $L_p[0,1]$ (the space of all measurable functions f on $[0,1]$ for which $\int_0^1 |f|^p < \infty$ with $0 < p < 1$), Bastero [3] proved that no of its subspaces can be minimal. Furthermore, it is known that every quotient of $L_p[0,1]$ contains a basic sequence (see Kalton [14]). We refer also the reader to Tam [25] on the existence of basic sequences in complex quasi-Banach spaces with plurisubharmonic quasi-norms.

Contrary to previous results on the subject, we do not assume X to be complete nor to contain closed infinite dimensional subspaces with a separating family of continuous linear functionals. In particular, this provides a positive counterpart to the results of Kalton and Shapiro [13], who were the first persons to tackle the issue of the non-minimality of F-spaces under this kind of assumption. Notice nevertheless that the point-separating assumption in the main result above sounds like a natural hypothesis. Some of the reasons are as follows. Recall a topological vector space X is said to have the Hahn-Banach extension property (HBEP, in short) if every continuous linear functional defined on a closed subspace of X can be extended to a continuous linear functional on the whole space. It is easy to show that if X has the HBEP, then it has a separating dual space. It is clear that every metrizable locally convex space is non-minimal. Nevertheless there exist non-locally convex metric linear spaces with the HBEP (cf. [15, Theorem 1.3]). Finally, we observe from ([11] Proposition 2.2) that every subspace of ω is locally convex and hence has separating dual. We point out that differently from these and possibly other existing papers addressing similar issues, our approach for proving the main result above goes beyond some of the generally used since completeness is not a crucial ingredient here.

Next, we deal with some of the implications of our main result. In [20, Corollary 1.6] Peck proved that a metrizable topological vector space is locally convex iff all its $\aleph_0$-dimensional subspaces have the same property. Thus, we get the following.

Corollary 1.1. *Let X be a quasi-normed space with a separating dual space. Then X admits a strictly weaker Hausdorff non-locally convex vector topology.*

Remark 1. The problem of finding strictly weaker non-locally convex (locally bounded and dual-less) topologies on a given non dual-less metrizable topological vector space has been studied by Kąkol in [9, Propositon 5 and corollaries 6-7].

With regarding the basic sequence problem, we conclude also the following generalization of a result of Kalton and Shapiro (as mentioned in abstract).

Theorem 1.2. *Let X be a quasi-Banach space. Then the following are equivalent:*

(a) *X contains a regular basic sequence;*
(b) *X contains a strongly regular M-basic sequence;*
(c) *X contains an infinite dimensional closed subspace which admits a continuous norm;*
(d) *X contains an infinite countable dimensional subspace with a separating dual;*
(e) *X is not minimal.*

Proof. The equivalences between (a), (b) and (c) are due Kalton and Shapiro [13, Theorem 3.2] which, in fact, were proved in the general framework of F-spaces. It is worth pointing out that (c) is due to Drewnowski [6, Proposition 3.1-(f)]. It is obvious that (c) implies (d). That (d) implies (e) follows from our main result. Finally, using again the result of Kalton and Shapiro we obtain the implication (e) $\Rightarrow$ (a). $\square$

Another immediate consequence of our main result is the following result which brings out an apparently new feature of minimal spaces.

Theorem 1.3. *Let X be a minimal quasi-Banach space. Then every $\aleph_0$-dimensional subspace of X fails to have a separating family of continuous linear functionals.*

Remark 2. In 1994 Kalton [15, Theorem 1.1] solved the famous question concerning the existence of minimal spaces other than ω. He constructed a minimal quasi-Banach space X with one-dimensional subspace L so that if Y_0 is a closed infinite-dimensional subspace of X, then $L \subset Y_0$. In particular, X contains no basic sequence. Thus there exist locally bounded F-spaces with no basic sequence. Such spaces are minimal. This is due to the fact that any minimal space that has a basis is isomorphic to ω. Thus, since ω is not locally bounded, a minimal locally bounded F-space contains no basic sequences.

This paper is organized as follows. In Section 2, we shall recall some well-known results from the topological vector spaces theory. In Section 3, we establish the existence of a special type of Hamel-Schauder basis for infinite countable dimensional subspaces of Hausdorff topological vector spaces. For notational convenience, it will be called ℓ_1-Hamel Schauder basis. We will show in Section 4 that ℓ_1-Hamel Schauder basis can be used for building weaker Hausdorff vector topologies in locally bounded spaces subject to certain prescribed conditions. Finally, in last section we shall give the proof of main result. The proof is a slightly more involved version of the proof of Theorem 1.1 in Barroso-Mota [2], even though it essentially relies on ideas and methods of Kalton (1971), Kalton and Shapiro (1976), Peck (1993), and Kokk and Żelazko (1995).

2. Notations and background on topological vector spaces

Let us start by recalling some background on topological vector spaces, which can be found for instance in [24, 26]. Let X be a real linear space. A function $\|\cdot\|\colon X \to [0,\infty)$ satisfying:

(i) $\|x+y\| \leq \|x\| + \|y\|$ for all x, y in X;

(ii) $\|\alpha x\| \leq \|x\|$ for all $\alpha \in \mathbb{R}$, $|\alpha| \leq 1$ and $x \in X$;

(iii) $\|\alpha x\| \to 0$ as $\alpha \to 0$ in $\mathbb{R}$, whenever x is fixed in X

is called an F-seminorm. If in addition, we have $\|x\| = 0$ iff $x = 0$ then $\|\cdot\|$ is said to be an F-norm. An F-norm $\|\cdot\|$ is called a quasi-norm if it is homogeneous and there is a constant $C \geq 1$ such that

$$\|x+y\| \leq C(\|x\| + \|y\|),$$

for all $x, y \in X$. In this case $(X, \|\cdot\|)$ is said to be a quasi-normed space and the constant C is called the modulus of concavity of $\|\cdot\|$ (see [17, Chapter 25]). As it is well-known, every family $\mathcal{F}$ of F-seminorms on X determines a vector topology τ on X. In this case, a basis of τ-neighborhoods of zero in X consists of sets of the form

$$\{x \in X\colon \|x\|_{\alpha_i} < \epsilon,\, i = 1, 2, \ldots, k\},$$

where ϵ is an arbitrary positive number and $\|\cdot\|_{\alpha_1}, \|\cdot\|_{\alpha_2}, \ldots, \|\cdot\|_{\alpha_k}$ is any finite subcollection of $\mathcal{F}$. The converse is also well-known (see [26, Chapter 1, Proposition 2]): A linear topology on X can always be determined by a family of F-seminorms. In particular, every metrizable linear topology τ may be defined by a single F-norm. Conversely, if τ is Hausdorff and has a countable neighborhood basis of zero then it is metrizable. In such a case, the F-norm defines a translation invariant metric ρ and (X, ρ) is said to be a linear metric space. A complete linear metric space is called an F-space. In terms of 0-neighborhoods, the following result holds true (see [4]).

Proposition 2.1. *Let (X, τ) be a Hausdorff topological vector space, and let $\mathcal{B}(\tau)$ be a neighborhood basis of 0 for τ. Then the following properties hold true:*

(τ_1) *if $U, V \in \mathcal{B}(\tau)$, then there is a $W \in \mathcal{B}(\tau)$ such that $W \subset U \cap V$;*

(τ_2) *if $U \in \mathcal{B}(\tau)$ and $|t| \leq 1$, then $tU \subset U$;*

(τ_3) *if $U \in \mathcal{B}(\tau)$ there exists $V \in \mathcal{B}$ so that $V + V \subset U$;*

(τ_4) *0 is a core point of each U in $\mathcal{B}(\tau)$;*

(τ_5) *τ is Hausdorff if and only if 0 is the only point common to all U in $\mathcal{B}(\tau)$.*

Remark 3. Recall a point $x \in X$ is said to be a core point of a nonempty set A in X, if for every point $y \in X$ there is $\epsilon(y) > 0$ so that $x + ty \in A$ for all $|t| < \epsilon(y)$.

Conversely, if a neighborhood basis at 0 is chosen to satisfy (τ_1)–(τ_5) above, then it induces a Hausdorff vector topology on X. A vector topology τ is said to be locally convex, if a basis of neighborhoods of zero $\mathcal{B}(\tau)$ can be chosen so that every U in $\mathcal{B}(\tau)$ is convex. Recall the following characterization: τ is locally convex if and only if it is induced by a family $\mathcal{F}$ of seminorms on X.

Let (X,τ) be a topological vector space. A subset $A \subset X$ is said bounded if for every 0-neighborhood U there exists a scalar $\lambda \neq 0$ so that $\lambda A \subset U$. The space (X,τ) is said to be locally bounded if there exists a bounded neighborhood V of zero in X. Every quasi-normed space is locally bounded. An F-seminorm $\|\cdot\|$ on X is said to be p-homogeneous if $\|\lambda x\| = |\lambda|^p\|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in X$. The following result due Aoki [1] and Rolewicz [23] yields a characterization for locally bounded spaces.

Proposition 2.2. *A topological vector space (X,τ) is locally bounded if and only if there exists $0 < p < 1$ and a p-homogeneous F-norm determining a topology equivalent to the given one.*

We finish this section with the following additional notation: if F is a subspace of (X,τ), we denote by $\tau|_F$ the topology on F induced by τ.

3. ℓ_1-Hamel Schauder basis in spaces of countable dimension

In this section we are concerned with the existence of a special type of Hamel-Schauder basis for infinite countable dimensional subspaces of topological vector spaces. As we shall see, it will be an important ingredient in the proof of main result. Throughout this section, unless specified otherwise, (X,τ) stands for a topological vector space (TVS, in short) and $(X,\tau)^*$ its topological dual. Recall that if $(e_i)_i$ a sequence in X and (f_i) is a sequence in $(X,\tau)^*$, then $(e_i; f_i)_{i\in\mathbb{N}}$ is said to be a biorthogonal system if $f_i(e_j) = \delta_{ij}$. Given $x \in X$ the support of x is the set $\operatorname{supp}(x) = \{i\colon f_i^*(x) \neq 0\}$.

Definition 3.1. *Let (X,τ) be a TVS and $\mathbb{E}$ an infinite-dimensional subspace of X. Let (e_i) be a sequence in $\mathbb{E}$ and (e_i^*) be a sequence in $(\mathbb{E},\tau)^*$. Then, $(e_i; e_i^*)_{i\in\mathbb{N}}$ is said to be an almost-biorthogonal system for $\mathbb{E}$, if $e_i^*(e_j) = 0$ for all $i \neq j$ and $e_i^*(e_i) > 0$.*

A family of vectors $\{e_\alpha\colon \alpha \in \Lambda\}$ is said to be a Hamel basis for X, if every vector of X can be uniquely written as a linear combination of finitely many e_α's. In this case, to each $x \in X$ there corresponds a unique sequence of scalars $(f_i^*(x))_i$ such that the series $\sum_i f_i^*(x)e_i$ equals to x. If furthermore each f_i^* belongs to $(X,\tau)^*$, then (e_i) is called a Hamel-Schauder basis for X.

Definition 3.2. *Let C be a bounded subset of a TVS (X,τ) with $0 \in C$, and $\|\cdot\|$ a continuous F-seminorm on X. Assume that $\mathbb{E}$ is a countable dimensional subspace of X with the topology inherited from X. A ℓ_1-Hamel Schauder basis for $\mathbb{E}$ relative to the couple $(C,\|\cdot\|)$ is an almost-biorthogonal system $(e_i; e_i^*)_{i\in\mathbb{N}}$ for $\mathbb{E}$ such that (e_i) is a Hamel basis for $\mathbb{E}$, and for some B_{ℓ_1}-sequence of positive real numbers $(r_i)_{i\in\mathbb{N}}$, the following inequalities hold for each $i \in \mathbb{N}$*

$$2^i\|e_i^*\|_C\|e_i\| \leq r_i, \tag{1}$$

$$\|e_i^*\|_C \geq 1, \tag{2}$$

where $\|e_i^*\|_C = \sup_{z\in C\cap\mathbb{E}} |e_i^*(z)|$.

The following basic result is a direct consequence of Hahn-Banach theorem. It will be useful to prove our main result.

Lemma 3.1. *Let (X,τ) be a Hausdorff topological vector space and let $\mathbb{E}$ be an infinite countable dimensional subspace of X with a separating dual. Assume that C is a bounded subset of X such that 0 is a core point of C. Then for each p-homogeneous continuous F-seminorm $\|\cdot\|$ on X with $0<p<1$, there exists a ℓ_1-Hamel Schauder basis for $\mathbb{E}$ with respect to the couple $(C,\|\cdot\|)$.*

Proof. Write $\mathbb{E}=\bigcup_n \mathbb{E}_n$, where $(\mathbb{E}_n)_n$ is a sequence of n-dimensional subspaces of $\mathbb{E}$ so that $\mathbb{E}_n\subset\mathbb{E}_{n+1}$ for all $n\in\mathbb{N}$. Set $\mathbb{E}^*=(\mathbb{E},\tau|_{\mathbb{E}})^*$. Let now $(r_i)_{i\in\mathbb{N}}$ be a fixed sequence of positive real numbers in B_{ℓ_1}. Fix $a>0$ and pick $-a<b<-pa$. Next, we shall construct by induction a biorthogonal system $(e_i;e_i^*)_{i\in\mathbb{N}}$ in $\mathbb{E}\times\mathbb{E}^*$ having properties described above. The procedure for that is standard (see, for instance, Kalton [11, Proposition 1.1]). Initially, we will study the case $i=1$. Pick any $x_1\in\mathbb{E}_1\setminus\{0\}$ and choose (via Hahn-Banach theorem) a functional x_1^* in $\mathbb{E}^*$ so that $x_1^*(x_1)=1$. Put $y_s=s^a x_1$ and $y_s^*=s^b x_1^*$, $s>0$. Observe that

$$\begin{cases} y_s^*(y_s)=s^{a+b}\\ 2\|y_s^*\|_C\|y_s\|=2s^{ap+b}\|x_1^*\|_C\|x_1\|. \end{cases}$$

Thus, since $a+b>0$ and $ap+b<0$, we can obtain $s_1>0$ large enough that $y_{s_1}^*(y_{s_1})\geq 1$ and $2\|y_{s_1}^*\|_C\|y_{s_1}\|\leq r_1$. By the core assumption on C, we can select a number $t_1>0$ so that $t_1 y_{s_1}\in C\cap\mathbb{E}$. If we define $e_1=t_1^{1/p}y_{s_1}$ and $e_1^*=t_1^{-1}y_{s_1}^*$, then we have proved the first step of induction. Notice that $t_1y_{s_1}\in C\cap\mathbb{E}$ implies $\|e_1^*\|_C\geq 1$.

Let now $k\in\mathbb{N}$ be arbitrary and assume that finite sequences $(e_i)_{i=1}^k$ and $(e_i^*)_{i=1}^k$ have been chosen so that:

(i) $(e_i)_{i=1}^k$ is a Hamel basis for $\mathbb{E}_k$,

(ii) $(e_i^*)_{i=1}^k$ lies in $\mathbb{E}^*$, $e_i^*(e_j)=0$ if $i\neq j$ and $e_i^*(e_i)>0$,

(iii) $2^i\|e_i^*\|_C\|e_i\|\leq r_i$, and

(iv) $\|e_i^*\|_C\geq 1$,

hold true for all $1\leq i,j\leq k$. Since $\dim(\mathbb{E}_{k+1})=k+1$, there is $x_{k+1}\in\mathbb{E}_{k+1}\setminus\mathbb{E}_k$ so that $e_i^*(x_{k+1})=0$ for every $i=1,2,\dots,k$. Using again Hahn-Banach theorem (cf. [24, pg. 22, 3.3, and pg. 49, 4.2]), we find x_{k+1}^* in $\mathbb{E}^*$ so that $x_{k+1}^*(x_{k+1})=1$ and $x_{k+1}^*(e_i)=0$ for all $i\leq k$. Now put $z_s=s^a x_{k+1}$ and $z_s^*=s^b x_{k+1}^*$. Using then the same arguments as in the first step of the induction process, we can select numbers $s_{k+1},t_{k+1}>0$ such that $t_{k+1}z_{s_{k+1}}\in C\cap\mathbb{E}$, $z_{s_{k+1}}^*(z_{s_{k+1}})\geq 1$ and if

$$e_{k+1}=t_{k+1}^{1/p}z_{s_{k+1}} \ \text{ and } \ e_{k+1}^*=t_{k+1}^{-1}z_{s_{k+1}}^*$$

then $\|e_{k+1}^*\|_C\geq 1$ and

$$2^{k+1}\|e_{k+1}^*\|_C\|e_{k+1}\|\leq r_{k+1}.$$

This concludes the $(k+1)^{\text{th}}$-step of induction process. Clearly $(e_i)_{i\in\mathbb{N}}$ is a Hamel basis for $\mathbb{E}$ and $(e_i;e_i^*)_{i\in\mathbb{N}}$ defines a ℓ_1-Hamel Schauder basis for $\mathbb{E}$ relative to the couple $(C,\|\cdot\|)$. The proof is complete. $\square$

4. Weaker Hausdorff vector topologies in locally bounded spaces

Throughout this section, (X,τ) stands for a Hausdorff topological vector space whose topology is induced by a p-homogeneous $(0<p<1)$ F-norm $\|\cdot\|_p$ with $0<p<1$. Let C be a bounded closed subset of X with $0\in C$. Assume also that the diameter of C is positive. Given a countable dimensional subspace $\mathbb{E}$ of X, a ℓ_1-Hamel Schauder basis $(e_i;e_i^*)_{i\in\mathbb{N}}$ for $\mathbb{E}$ induces a natural norm $\|\cdot\|_{\mathbb{E}}$ given by

$$\|x\|_{\mathbb{E}}=\sum_{n=1}^{\infty}\frac{|e_n^*(x)|}{2^n\|e_n^*\|_C},\quad x\in\mathbb{E},$$

where each $\|e_i^*\|_C$ is as in Lemma 3.1. Our next result provides a criterion for extending the relative topology on $\mathbb{E}$ inherited from X to one on the whole space being weaker than the original. As we will see, the norm $\|\cdot\|_{\mathbb{E}}$ will play an important role in the construction process of the new topology.

Theorem 4.1. *Let X, C, and $\|\cdot\|_p$ be as above. Assume that $\mathbb{E}$ is an infinite countable dimensional subspace of X with a ℓ_1-Hamel Schauder basis $(e_i;e_i^*)_{i\in\mathbb{N}}$ with respect to the couple $(C,\|\cdot\|_p)$. Then there exists a weaker Hausdorff vector topology ρ on X such that $x_n\to 0$ in (X,ρ) whenever $(x_n)_n$ is a sequence in $(C-C)\cap\mathbb{E}$ so that $\|x_n\|_{\mathbb{E}}\to 0$.*

For the proof we will need the following lemma.

Lemma 4.2. *Let $(\lambda_i)_{i=1}^N$, $(t_i)_{i=1}^\infty$ and $(a_i)_{i=1}^\infty$ be sequences of positive real numbers with $(a_i)_{i=1}^\infty\in\ell_1$. Assume that $p_2\le p_1$ are in $(0,1)$. Then the following inequality holds true:*

$$\left(\sum_{i=1}^N\lambda_i^{p_1}t_ia_i^{1-p_2}\right)^{1/p_1}\left(\sum_{i=1}^\infty a_i\right)^{1/q_1}\le\sum_{i=1}^N\lambda_it_i^{1/p_1}a_i^{\frac{p_1-p_2}{p_1}},$$

where p_1 and q_1 are conjugate numbers, that is, $1/p_1+1/q_1=1$.

Proof. This is a consequence of Hölder's inequality, which we include here for the sake of completeness. Let $C_i=t_ia_i^{1-p_2}$ with $i\in\mathbb{N}$, and consider real functions $f,g\colon\mathbb{N}\to\mathbb{R}_+$ given by

$$f(i)=\begin{cases}\lambda_i & \text{if } i=1,\dots,N,\\ 0 & \text{if } i>N,\end{cases}$$

and

$$g(i)=\left(a_iC_i^{-1}\right)^{1/q_1},$$

for all i in $\mathbb{N}$. We then define a discrete measure μ on $\mathbb{N}$ by $\mu(i)=C_i$ for all $i\ge 1$. From the reverse Hölder inequality (cf. [19]):

$$\left(\int_{\mathbb{N}}f(i)^{p_1}d\mu\right)^{1/p_1}\left(\int_{\mathbb{N}}g(i)^{q_1}d\mu\right)^{1/q_1}\le\int_{\mathbb{N}}f(i)g(i)d\mu,$$

we see that

$$\left(\sum_{i=1}^{N}\lambda_i^{p_1}t_i a_i^{1-p_2}\right)^{1/p_1}\left(\sum_{i=1}^{\infty}a_i\right)^{1/q_1}\leq\sum_{i=1}^{N}\lambda_i t_i^{1/p_1}a_i^{\frac{p_1-1}{p_1}}a_i^{\frac{1-p_2}{p_1}},$$

which in turn proves the result. □

We are now ready to start the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $K = C - C$. We start by observing that $\|\cdot\|_{\mathbb{E}}$ fulfills the following property: for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ so that

$$z \in (K - K)\cap\mathbb{E} \text{ and } \|z\|_p < \delta(\epsilon) \text{ imply } \|z\|_{\mathbb{E}} < \epsilon. \tag{3}$$

This can be easily verified by observing the fact that the boundedness of C implies that the sequence of partial sums $\{\|\cdot\|_{\mathbb{E}}^N\}_{N\in\mathbb{N}}$ given by

$$\|x\|_{\mathbb{E}}^N = \sum_{n=1}^{N}\frac{|e_n^*(x)|}{2^n\|e_n^*\|_C},$$

converge uniformly to $\|x\|_{\mathbb{E}}$ with respect to x in $(K-K)\cap\mathbb{E}$. Let now $\mathcal{S}$ be a subset of the space $\mathbb{R}^{\mathbb{N}}$ given by

$$\mathcal{S} = \{\mathfrak{a}\colon \mathbb{N}\to\mathbb{R}\colon 0 < \mathfrak{a}(n) < 1 \text{ for all } n\in\mathbb{N}\}.$$

Define subsets $F_{(n,\mathfrak{a})}$ of X as follows: for each n in $\mathbb{N}$ and $\mathfrak{a}$ in $\mathcal{S}$, put

$$F_{(n,\mathfrak{a})} = \overline{G_{(n,\mathfrak{a})}}^{\tau},$$

where

$$G_{(n,\mathfrak{a})} = \left\{z\in K\cap\mathbb{E}:\ \|z\|_{(n,\mathfrak{a})}\leq\frac{1}{2^n}\right\}$$

and

$$\|z\|_{(n,\mathfrak{a})} = \sup_{\mathfrak{b}\in[\mathbb{N}]^n}\sum_{i=1}^{n}\left(\frac{|e^*_{\mathfrak{b}(i)}(z)|}{2^{\mathfrak{b}(i)}\|e^*_{\mathfrak{b}(i)}\|_C\exp(\mathfrak{a}(i))}\right)^{\mathfrak{a}(i)},\quad z\in\mathbb{E}.$$

Here $[\mathbb{N}]^n$ denotes the set of all finite subsets of the positive integers with cardinality equal to n, and $\mathfrak{b}\in[\mathbb{N}]^n$ is written as $\mathfrak{b} = (\mathfrak{b}(1),\mathfrak{b}(2),\dots,\mathfrak{b}(n))$ with $\mathfrak{b}(1)<\mathfrak{b}(2)<\cdots<\mathfrak{b}(n)$. Following Peck [22, Lemma 1], we define a vector topology ρ on X which is weaker than τ and has as base of neighborhoods of 0 sets of the form $F_{(n,\mathfrak{a})}+U$, where U runs over all the τ-neighborhoods of 0 in X; one readily verifies (τ_1)–(τ_4) of Proposition 2.1. Now in order to show that ρ is Hausdorff, we will follow the approach of Kalton-Shapiro [13, Proposition 3.1]. Firstly, observe that since each $F_{(n,\mathfrak{a})}$ is τ-closed in X we get

$$\begin{aligned}\bigcap_{(n,\mathfrak{a}),U}(F_{(n,\mathfrak{a})}+U) &= \bigcap_{(n,\mathfrak{a})}\bigcap_{U}\big(F_{(n,\mathfrak{a})}+U\big)\\ &= \bigcap_{(n,\mathfrak{a})}F_{(n,\mathfrak{a})}.\end{aligned}$$

Thus it suffices to prove that if $\mathcal{Q} \in F_{(n,\mathfrak{a})}$ for all $n \in \mathbb{N}$ and every $\mathfrak{a} \in \mathcal{S}$, then $\mathcal{Q} = 0$. Suppose by contradiction that $\mathcal{Q}$ lies within all the sets $F_{(n,\mathfrak{a})}$, but $\mathcal{Q} \neq 0$. Next, we shall use methods from the paper of Kokk and Żelazko [18] to prove the two claims below. These statements will help us to conclude that $\mathcal{Q} = 0$. Thus, producing a contradiction which will imply that ρ must be Hausdorff. For each $n \in \mathbb{N}$ and $\mathfrak{a} \in \mathcal{S}$, let $(x_\sigma^{(n,\mathfrak{a})})_\sigma$ be a sequence in $G_{(n,\mathfrak{a})}$ converging to p with respect to τ.

Claim 1. There exists $(n,\mathfrak{a}) \in \mathbb{N} \times \mathcal{S}$ such that only finite many indices i do satisfy

$$\lim_\sigma e_i^*(x_\sigma^{(n,\mathfrak{a})}) \neq 0.$$

Suppose by contradiction that this is not the case. Put for $i \in \mathbb{N}$

$$a_i^{(n,\mathfrak{a})} = \lim_\sigma d_i e_i^*(x_\sigma^{(n,\mathfrak{a})}),$$

where $d_i = 1/2^i \|e_i^*\|_C$. Thus for each pair $(n,\mathfrak{a}) \in \mathbb{N} \times \mathcal{S}$ there exists an increasing sequence $(i_k^{(n,\mathfrak{a})})_{k\in\mathbb{N}}$ of positive integers so that

$$a_{i_k^{(n,\mathfrak{a})}}^{(n,\mathfrak{a})} \neq 0, \text{ for all } k \in \mathbb{N}.$$

We distinguish now between two mutually exclusive cases:

Case 1. There is $(n,\mathfrak{a}) \in \mathbb{N} \times \mathcal{S}$ so that

$$\inf_{k\in\mathbb{N}} \left|a_{i_k^{(n,\mathfrak{a})}}^{(n,\mathfrak{a})}\right| \geq \eta > 0, \text{ for some } \eta > 0.$$

If this is the case, then since $(x_\sigma^{(n,\mathfrak{a})})_\sigma$ is τ-Cauchy from (3) we conclude that

$$\limsup_\sigma \|x_\sigma^{(n,\mathfrak{a})}\|_{\mathbb{E}} < \infty.$$

Thus, we get

$$N \leq \frac{1}{\eta} \limsup_\sigma \|x_\sigma^{(n,\mathfrak{a})}\|_{\mathbb{E}}, \quad \text{for all } N \geq 1.$$

This contradiction eliminates this case.

Case 2. The contrary case does not hold, too. Indeed, if

$$\liminf_k \left|a_{i_k^{(n,\mathfrak{a})}}^{(n,\mathfrak{a})}\right| = 0 \text{ for all } (n,\mathfrak{a}) \in \mathbb{N} \times \mathcal{S}$$

then, by passing to a further subsequence, if needed, we can assume that each sequence $(i_k^{(n,\mathfrak{a})})_{k\in\mathbb{N}}$ is such that

$$\left|a_{i_k^{(n,\mathfrak{a})}}^{(n,\mathfrak{a})}\right| < 1, \text{ for all } k.$$

Define a new function $\mathfrak{u}_{(n,\mathfrak{a})} \colon \mathbb{N} \to \mathbb{R}_+$ in $\mathcal{S}$ by putting

$$\mathfrak{u}_{(n,\mathfrak{a})}(k) = \left|a_{i_k^{(n,\mathfrak{a})}}^{(n,\mathfrak{a})}\right|, \quad k \in \mathbb{N}.$$

We then obtain the following estimate

$$\begin{aligned}
\sum_{k=1}^{n}\Big(\frac{\mathfrak{u}_{(n,\mathfrak{a})}(k)}{\exp(\mathfrak{u}_{(n,\mathfrak{a})}(k))}\Big)^{\mathfrak{u}_{(n,\mathfrak{a})}(k)} &= \lim_{\sigma}\sum_{k=1}^{n}\left(\frac{d_{i_k^{(n,\mathfrak{a})}}\Big|e^*_{i_k^{(n,\mathfrak{a})}}(x_\sigma^{(n,\mathfrak{a})})\Big|}{\exp(\mathfrak{u}_{(n,\mathfrak{a})}(k))}\right)^{\mathfrak{u}_{(n,\mathfrak{a})}(k)}\\
&\leq \lim_{\sigma}\sup_{\mathfrak{b}\in[\mathbb{N}]^n}\sum_{k=1}^{n}\left(\frac{d_{\mathfrak{b}(k)}\Big|e^*_{\mathfrak{b}(k)}(x_\sigma^{(n,\mathfrak{a})})\Big|}{\exp(\mathfrak{u}_{(n,\mathfrak{a})}(k))}\right)^{\mathfrak{u}_{(n,\mathfrak{a})}(k)}\\
&\leq \frac{1}{2^n}+\lim_{\sigma}\Delta(n,\mathfrak{a},\sigma),
\end{aligned}$$

for all $n\in\mathbb{N}$ and $k=1,2,\dots,n$, where

$$\Delta(n,\mathfrak{a},\sigma)=\sup_{\mathfrak{b}\in[\mathbb{N}]^n}\sum_{k=1}^{n}\left(\frac{d_{\mathfrak{b}(k)}\Big|e^*_{\mathfrak{b}(k)}\Big(x_\sigma^{(n,\mathfrak{a})}-x_\sigma^{(n,\mathfrak{u}_{(n,\mathfrak{a})})}\Big)\Big|}{\exp(\mathfrak{u}_{(n,\mathfrak{a})}(k))}\right)^{\mathfrak{u}_{(n,\mathfrak{a})}(k)}.$$

Using again property (3) and now the fact that $\Big(x_\sigma^{(n,\mathfrak{a})}-x_\sigma^{(n,\mathfrak{u}_{(n,\mathfrak{a})})}\Big)_\sigma$ converges to 0 with respect to τ, we can conclude that

$$\lim_{\sigma}\Delta(n,\mathfrak{a},\sigma)=0,$$

for all $n\in\mathbb{N}$ and every $\mathfrak{a}\in\mathcal{S}$. Hence, for a fixed $\mathfrak{a}$ in $\mathcal{S}$, this implies that the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ given by

$$\mathcal{S}_n=\Big(\frac{\mathfrak{u}_{(n,\mathfrak{a})}(1)}{\exp(\mathfrak{u}_{(n,\mathfrak{a})}(1))}\Big)^{\mathfrak{u}_{(n,\mathfrak{a})}(1)},$$

converges to zero as n goes to infinity. But this is impossible since

$$\log\mathcal{S}_n=\mathfrak{u}_{(n,\mathfrak{a})}(1)\log\mathfrak{u}_{(n,\mathfrak{a})}(1)-\mathfrak{u}^2_{(n,\mathfrak{a})}(1),$$

and $0<\mathfrak{u}_{(n,\mathfrak{a})}(k)\leq 1$ for all $n,k\in\mathbb{N}$.

Claim 2. $\mathcal{Q}\in(C-C)\cap\mathbb{E}$. Indeed, by **Claim 1** there exist a pair $(n,\mathfrak{a})\in\mathbb{N}\times\mathcal{S}$ and a finite set $I^{(n,\mathfrak{a})}\subset\mathbb{N}$ so that

$$a_i^{(n,\mathfrak{a})}=0\quad\text{for all } i\notin I^{(n,\mathfrak{a})}.$$

Let

$$x^{(n,\mathfrak{a})}=\sum_{i\in I^{(n,\mathfrak{a})}}\big(\lim_{\sigma}e_i^*(x_\sigma^{(n,\mathfrak{a})})\big)e_i,$$

and consider the sequence $(y_\sigma^{(n,\mathfrak{a})})_\sigma$ given by $y_\sigma^{(n,\mathfrak{a})}=x_\sigma^{(n,\mathfrak{a})}-x^{(n,\mathfrak{a})}$. It suffices to show then that

$$y_\sigma^{(n,\mathfrak{a})}\to 0\ \text{ in } (X,\tau).$$

The proof of this depends on a suitable adaptation of the support technique employed in [18] to our context. Differently from there, here there is a little obstacle because $\|\cdot\|_p$ is not necessarily homogeneous. We overcome this difficulty by using that reverse Hölder's

inequality given in Lemma 4.2. Suppose then by contradiction that $(y_\sigma^{(n,\mathfrak{a})})_\sigma$ does not converge to zero in X. It follows that

$$M_{(n,\mathfrak{a})}\colon = \lim_\sigma \|y_\sigma^{(n,\mathfrak{a})}\|_p > 0.$$

For technical reasons, we make a strategic pause to list some properties of $(y_\sigma^{(n,\mathfrak{a})})_\sigma$:

(A) $\lim_\sigma e_i^*(y_\sigma^{(n,\mathfrak{a})}) = 0$ for all $i \in \mathbb{N}$.
(B) Given $0 < \epsilon < 1$, there exists $\sigma_{(\epsilon,n,\mathfrak{a})}$ so that $\|y_\sigma^{(n,\mathfrak{a})} - y_\nu^{(n,\mathfrak{a})}\|_p < \delta(\epsilon)$ for all $\sigma, \nu \geq \sigma_{(\epsilon,n,\mathfrak{a})}$, where $\delta(\epsilon)$ is as in (3).

The proof of (A) is similar to that in [18]. While (B) follows from the fact that $(y_\sigma^{(n,\mathfrak{a})})_\sigma$ is τ-Cauchy. Let now $\nu \geq \sigma_{(\epsilon,n,\mathfrak{a})}$ be fixed and put

$$J_\nu^{(n,\mathfrak{a})} = \operatorname{supp}(y_\nu^{(n,\mathfrak{a})}).$$

Following [18], we define the projection mapping $P_\nu^{(n,\mathfrak{a})}\colon \mathbb{E} \to \mathbb{E}$ by

$$P_\nu^{(n,\mathfrak{a})}(x) = \sum_{i \in J_\nu^{(n,\mathfrak{a})}} e_i^*(x) e_i.$$

From (A) we have

$$\lim_\sigma \sum_{i \in J_\nu^{(n,\mathfrak{a})}} \|e_i^*(P_\nu^{(n,\mathfrak{a})}(y_\sigma^{(n,\mathfrak{a})}))e_i\|_p = 0.$$

Fix now any $0 < \epsilon < 1$. Informations (3) and (B) imply that

$$\|y_\sigma^{(n,\mathfrak{a})} - y_\nu^{(n,\mathfrak{a})}\|_{\mathbb{E}} < \epsilon, \quad \text{for all } \sigma, \nu \geq \sigma_{(\epsilon,n,\mathfrak{a})}. \tag{4}$$

Let us now select a subsequence $(y_{\sigma_\alpha}^{(n,\mathfrak{a})})_\alpha$ of $(y_\sigma^{(n,\mathfrak{a})})_\sigma$ (with $\sigma_\alpha \geq \sigma_{(\epsilon,n,\mathfrak{a})}$ for all α) so that for every α

$$\frac{M_{(n,\mathfrak{a})}}{2} < \|y_{\sigma_\alpha}^{(n,\mathfrak{a})}\|_p.$$

Let I denote the identity operator on $\mathbb{E}$, and for $\nu \geq \sigma_{(\epsilon,n,\mathfrak{a})}$ put

$$J_{\nu,\alpha}^{(n,\mathfrak{a})}\colon = \operatorname{supp}((I - P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})})).$$

A simple computation shows that

$$y_{\sigma_\alpha}^{(n,\mathfrak{a})} = P_\nu^{(n,\mathfrak{a})}\big(y_{\sigma_\alpha}^{(n,\mathfrak{a})}\big) + \sum_{i \in J_{\nu,\alpha}^{(n,\mathfrak{a})}} e_i^*\Big(\big(I - P_\nu^{(n,\mathfrak{a})}\big)\big(y_{\sigma_\alpha}^{(n,\mathfrak{a})}\big)\Big)e_i.$$

Applying the triangular inequality and using the ℓ_1-property of the system $(e_i; e_i^*)_{i\in\mathbb{N}}$, we obtain that

$$\begin{aligned}
\frac{M_{(n,\mathfrak{a})}}{2} &< \|P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\|_p + \sum_{i\in J_{\nu,\alpha}^{(n,\mathfrak{a})}} \left|e_i^*((I-P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))\right|^p \|e_i\|_p \\
&< \|P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\|_p^+ \sum_{i\in J_{\nu,\alpha}^{(n,\mathfrak{a})}} \left|e_i^*((I-P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))\right|^p d_i r_i \\
&< \|P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\|_p + \sum_{i\in J_{\nu,\alpha}^{(n,\mathfrak{a})}} \left|e_i^*((I-P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))\right|^p \left(\frac{1}{2^i\|e_i^*\|_C}\right) r_i^{1-p}. \qquad (5)
\end{aligned}$$

Let us remember that the ℓ_1-property of the system $(e_i; e_i^*)$ states that

$$\|e_i\|_p \le d_i r_i,$$

where $d_i = 1/2^i\|e_i^*\|_C$ and $(r_i) \in \ell_1$ with $0 < r_i < 1$ for all $i \in \mathbb{N}$. Now we would like to establish a key estimative for the second summand on the right-hand side of (5) in terms of $\|\cdot\|_{\mathbb{E}}$. To furnish it, we define a function $|\cdot|_{(p)} : \mathbb{E} \to [0,\infty)$ as follows: if $x \in \mathbb{E}$ then

$$|x|_{(p)} = \sum_{i\in\mathbb{N}} |e_i^*(x)|^p \left(\frac{1}{2^i\|e_i^*\|_C}\right) r_i^{1-p}.$$

By applying Lemma 3.1 with $t_i = \left(\frac{1}{2^i\|e_i^*\|_C}\right)$, $p_1 = p$, $p_2 = p$ and $a_i = r_i$, we get for all $x \in \mathbb{E}$ that

$$\begin{aligned}
|x|_{(p)} &\le \|(r_i)_{i=1}^\infty\|_{\ell_1}^{1-p} \sum_{i=1}^\infty |e_i^*(x)| \left(\frac{1}{2^i\|e_i^*\|_C}\right)^{\frac{1}{p}}, \qquad (6)\\
&\le \|(r_i)_{i=1}^\infty\|_{\ell_1}^{1-p} \|x\|_{\mathbb{E}}^p.
\end{aligned}$$

At this point, we emphasize the importance of the property $\|e_i^*\|_C \ge 1$ in (1).

From [18] we know for all $x \in \mathbb{E}$ that

$$\operatorname{supp}(I - P_\nu^{(n,\mathfrak{a})})(x) \cap \operatorname{supp}(P_\nu^{(n,\mathfrak{a})}(x)) = \emptyset$$

and also

$$\operatorname{supp}((I - P_\nu^{(n,\mathfrak{a})})(x)) \cap J_\nu^{(n,\mathfrak{a})} = \emptyset.$$

Therefore,

$$\begin{aligned}
\left|y_{\sigma_\alpha}^{(n,\mathfrak{a})} - y_\nu^{(n,\mathfrak{a})}\right|_{(p)} &= \left|P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})}) - y_\nu^{(n,\mathfrak{a})} + (I - P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\right|_{(p)} \\
&= \left|P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})}) - y_\nu^{(n,\mathfrak{a})}\right|_{(p)} + \left|(I - P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\right|_{(p)}
\end{aligned}$$

and hence

$$\left|(I - P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\right|_{(p)} \le \left|y_{\sigma_\alpha}^{(n,\mathfrak{a})} - y_\nu^{(n,\mathfrak{a})}\right|_{(p)}. \qquad (7)$$

Using (4), (6) and (7) we can readily estimate for all α

$$\begin{aligned}
\frac{M_{(n,\mathfrak{a})}}{2} &< \sum_{i\in J_\nu^{(n,\mathfrak{a})}} \|e_i^*(P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))e_i\|_p + \left|(I-P_\nu^{(n,\mathfrak{a})})(y_{\sigma_\alpha}^{(n,\mathfrak{a})})\right|_{(p)} \\
&< \sum_{i\in J_\nu^{(n,\mathfrak{a})}} \|e_i^*(P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))e_i\|_p + |y_{\sigma_\alpha}^{(n,\mathfrak{a})} - y_\nu^{(n,\mathfrak{a})}|_{(p)} \\
&< \sum_{i\in J_\nu^{(n,\mathfrak{a})}} \|e_i^*(P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))e_i\|_p + \|(r_i)_{i=1}^\infty\|_{\ell_1}^{1-p}\|y_{\sigma_\alpha}^{(n,\mathfrak{a})} - y_\nu^{(n,\mathfrak{a})}\|_{\mathbb{E}}^p \\
&< \sum_{i\in J_\nu^{(n,\mathfrak{a})}} \|e_i^*(P_\nu^{(n,\mathfrak{a})}(y_{\sigma_\alpha}^{(n,\mathfrak{a})}))e_i\|_p + \|(r_i)_{i=1}^\infty\|_{\ell_1}^{1-p}\epsilon^p.
\end{aligned}$$

This yields a contradiction since $\epsilon > 0$ was arbitrary, and so proves the claim.

Let us turn now to the proof that ρ is Hausdorff. From **Claim 2** we know that $\mathfrak{Q}$ belongs to $(C-C)\cap\mathbb{E}$. Then for any fixed $m\in\mathbb{N}$, since

$$x_\sigma^{(m,\mathfrak{c})} \to \mathfrak{Q} \ \text{ in } \ (X,\tau) \ \text{ for all } \mathfrak{c}\in\mathcal{S},$$

we can also reach the conclusion that $\mathfrak{Q}$ is in $G_{(m,\mathfrak{c})}$ for all $\mathfrak{c}\in\mathcal{S}$. Indeed, taking into account that $x_\sigma^{(m,\mathfrak{c})}\in G_{(m,\mathfrak{c})}$ we get

$$\sum_{i=1}^m \left(\frac{|e_i^*(x_\sigma^{(m,\mathfrak{c})})|}{2^i\|e_i^*\|_C\exp(\mathfrak{c}(i))}\right)^{\mathfrak{c}(i)} \le \frac{1}{2^m}.$$

Using now that $\mathfrak{Q}\in\mathbb{E}$ and taking the limit on σ we obtain

$$\sum_{i=1}^m \left(\frac{|e_i^*(\mathfrak{Q})|}{2^i\|e_i^*\|_C\exp(\mathfrak{c}(i))}\right)^{\mathfrak{c}(i)} \le \frac{1}{2^m}$$

This implies that $\mathfrak{Q}=0$ because m was arbitrary. This concludes the proof that ρ is Hausdorff.

Finally, if (x_n) is a sequence in $K\cap\mathbb{E}$ so that $\|x_n\|_{\mathbb{E}}\to 0$, then using the neighborhoods $F_{(n,\mathfrak{a})}$ given in the construction of ρ it is easy to see that $x_n\to 0$ with respect to ρ. The proof of theorem is complete. □

5. Proof of main result

Let τ denote the topology of X and $\|\cdot\|$ a p-homogeneous F-seminorm determining τ, where $0<p<1$. Let $\mathbb{E}$ be an infinite countable dimensional subspace of X with

separating dual space, and let $(C_n)_{n=1}^{\infty}$ be an increasing sequence of infinite-dimensional bounded subsets of X such that $0 \in \mathrm{Int}(C_n)$ for all $n \geq 1$ and

$$X = \bigcup_{n=1}^{\infty} C_n.$$

Fix any $0 < \theta_p < p$. Given $n \in \mathbb{N}$ we use Lemma 3.1 to find an ℓ_1-Hamel Schauder basis $(e_{n,i}; e_{n,i}^*)_{i\in\mathbb{N}}$ for $\mathbb{E}$ relative to the triple $(C_n, \|\cdot\|, \theta_p)$. For each $n \in \mathbb{N}$, consider the natural norm $\|\cdot\|_n$ on $\mathbb{E}$ given by

$$\|z\|_n = \sum_{i=1}^{\infty} \frac{|e_{n,i}^*(z)|}{2^i \|e_{n,i}^*\|_n}, \quad z \in \mathbb{E}, \tag{8}$$

where $\|e_{n,i}^*\|_n = \sup_{z\in C_n\cap\mathbb{E}} |e_{n,i}^*(z)|$. As in (3), we can prove that if $K_n = C_n - C_n$, then for any $\epsilon > 0$ there exists $\delta(n,\epsilon) > 0$ so that

$$z \in (K_n - K_n) \cap \mathbb{E} \text{ and } \|z\|_p < \delta(n,\epsilon) \text{ imply } \|z\|_n < \epsilon.$$

Let us consider on $\mathbb{E}$ the locally convex topology $\mathcal{T}$ induced by the family $\{\|\cdot\|_n \colon n \in \mathbb{N}\}$, where each $\|\cdot\|_n$ is as in (8). Notice $\mathcal{T}$ is Hausdorff and metrizable. We claim that there exists $n_0 \in \mathbb{E}$ so that $\mathcal{T}|_{C_{n_0}\cap\mathbb{E}} < \tau|_{C_{n_0}\cap\mathbb{E}}$. It suffices to prove this for the case in which $\tau|_{\mathbb{E}}$ is locally convex, since otherwise the claim is trivial (for we would have $\mathcal{T} < \tau|_{\mathbb{E}}$). Let ω^τ be the weak topology induced by τ on $\mathbb{E}$. It is easy to see that $\mathcal{T}|_{C_n\cap\mathbb{E}} \leq \omega^\tau|_{C_n\cap\mathbb{E}}$ for every $n \in \mathbb{N}$. Since $\tau|_{\mathbb{E}}$ is locally convex, $(\mathbb{E},\tau)$ is a normed space and hence $\omega^\tau|_{\mathbb{E}} < \tau|_{\mathbb{E}}$. This implies that $\mathcal{T}|_{C_n\cap\mathbb{E}} < \tau|_{C_n\cap\mathbb{E}}$ for every $n \in \mathbb{N}$, and proves the claim. It follows, in particular, that there exists a sequence (x_k) in $K_{n_0} \cap \mathbb{E}$ so that $\|x_k\|_{n_0} \to 0$, but $x_k \not\to 0\,(\tau)$. By Theorem 4.1, if ρ denotes the weaker Hausdorff vector topology corresponding to $(C_{n_0}, \|\cdot\|, \theta_p)$ then we can conclude that $x_k \to 0$ in (X,ρ). In particular, $\rho < \tau$ and the proof of theorem is complete.

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